HW 13

Page 287

① let C be the unit circle and $z = e^{i\theta}$, $\theta \in (0, 2\pi]$.

$$\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = \int_0^{2\pi} \frac{ie^{i\theta}\,d\theta}{5+2(z+1/z)}\left(\frac{1}{ie^{i\theta}}\right)$$

$$= \int_0^{2\pi} \frac{dz}{\left(5z+2(z^2+1)\right)i}$$

$$2z^2+5z+2=0 \text{ iff } z=-2 \text{ or } -1/2$$

Thus, $$\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = \int_C \frac{dz}{2i(z+2)(z+1/2)}$$

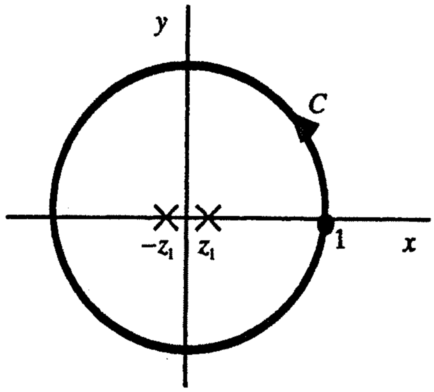
$$= \pi \frac{1}{3/2} = \frac{3\pi}{2}.$$

Page 287

2. To evaluate the definite integral in question, write

$$\int_{-\pi}^{\pi}\frac{d\theta}{1+\sin^2\theta}=\int_C\frac{1}{1+\left(\frac{z-z^{-1}}{2i}\right)^2}\cdot\frac{dz}{iz}=\int_C\frac{4iz\,dz}{z^4-6z^2+1},$$

where C is the positively oriented unit circle $|z|=1$. This circle is shown below.



Solving the equation $(z^2)^2-6(z^2)+1=0$ for z^2 with the aid of the quadratic formula, we find that the zeros of the polynomial z^4-6z^2+1 are the numbers z such that $z^2=3\pm 2\sqrt{2}$. Those zeros are, then, $z=\pm\sqrt{3+2\sqrt{2}}$ and $z=\pm\sqrt{3-2\sqrt{2}}$. The first two of these zeros are exterior to the circle, and the second two are inside of it. So the singularities of the integrand in our contour integral are

$$z_1=\sqrt{3-2\sqrt{2}} \quad \text{and} \quad z_2=-z_1,$$

indicated in the figure. This means that

$$\int_{-\pi}^{\pi}\frac{d\theta}{1+\sin^2\theta}=2\pi i(B_1+B_2),$$

where

$$B_1=\operatorname*{Res}_{z=z_1}\frac{4iz}{z^4-6z^2+1}=\frac{4iz_1}{4z_1^3-12z_1}=\frac{i}{z_1^2-3}=\frac{i}{(3-2\sqrt{2})-3}=-\frac{i}{2\sqrt{2}}$$

and

$$B_2=\operatorname*{Res}_{z=-z_1}\frac{4iz}{z^4-6z^2+1}=\frac{-4iz_1}{-4z_1^3+12z_1}=\frac{i}{z_1^2-3}=-\frac{i}{2\sqrt{2}}.$$

Since

$$2\pi i(B_1+B_2)=2\pi i\left(-\frac{i}{\sqrt{2}}\right)=\frac{2\pi}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}=\sqrt{2}\pi,$$

the desired result is

$$\int_{-\pi}^{\pi}\frac{d\theta}{1+\sin^2\theta}=\sqrt{2}\pi.$$

③ $\cancel{\frac{1+\cos 6\theta}{2}}$

$$\frac{\cos^2 3\theta}{5-4\cos 2\theta} = \frac{1+\cos 6\theta}{2(5-4\cos 2\theta)}$$

If $z = e^{i\theta}$, $$\frac{\cos^2 3\theta}{5-4\cos 2\theta} = \frac{1+\left(\frac{1}{z^6}+z^6\right)\frac{1}{2}}{2\left(5-2\left(\frac{1}{z^2}+z^2\right)\right)}$$

$$= \frac{z^{12}+2z^6+1}{4z^4(-2z^4+5z^2-2)}$$

The zeros of denominator inside the unit circle are

$z = \pm\frac{1}{\sqrt{2}}$,

$$\int_0^{2\pi} \frac{\cos^2 3\theta}{5-4\cos 2\theta}\, d\theta = \int_C \frac{z^{12}+2z^6+1}{4z^4(-2z^4+5z^2-2)} \cdot \frac{dz}{iz}$$

let $f(z) = \frac{z^{12}+2z^6+1}{z^5(-2z^4+5z^2-2)}$, by long division,

$$f(z) = -\frac{z^8}{2} - \frac{5z^6}{4} - \frac{21}{8}z^4 \cdots$$

Thus, $\operatorname*{Res}_{z=0} \frac{f(z)}{z^5} = \frac{-21}{8}$.

$\operatorname*{Res}_{z=\pm\frac{1}{\sqrt{2}}} \frac{f(z)}{z^5} = \frac{27}{16}$

$$\therefore \int_0^{2\pi} \frac{\cos^2 3\theta}{5-4\cos 2\theta}\, d\theta = \frac{1}{4i} \cdot 2\pi i \left(\frac{12}{16}\right) = \frac{3\pi}{8}.$$

(4) let $z = e^{i\theta}$, then

$$\frac{1}{1 + a\cos\theta} = \frac{1}{1 + a\left(z + \frac{1}{z}\right)\frac{1}{2}}$$
$$= \frac{2z}{az^2 + 2z + a}$$

The zeros of denominator are $z = -\frac{1}{a} \pm \sqrt{\frac{1}{a^2} - 1}$

Since $|a| < 1$, the only pole inside the unit circle is

$$z_0 = -\frac{1}{a} + \sqrt{\frac{1}{a^2} - 1}$$

$$\int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} = \int_0^{2\pi} \frac{2z}{az^2 + 2z + a} \frac{dz}{iz}$$

$$= \int_C \frac{-2i}{az^2 + 2z + a} dz$$

$$= -2i(2\pi i) \operatorname*{Res}_{z = z_0} \left(\frac{1}{az^2 + 2z + a}\right)$$

$$= 4\pi \left(\frac{1}{2a\sqrt{\frac{1}{a^2} - 1}}\right)$$

$$= \frac{2\pi}{\sqrt{1 - a^2}}.$$

Page 293

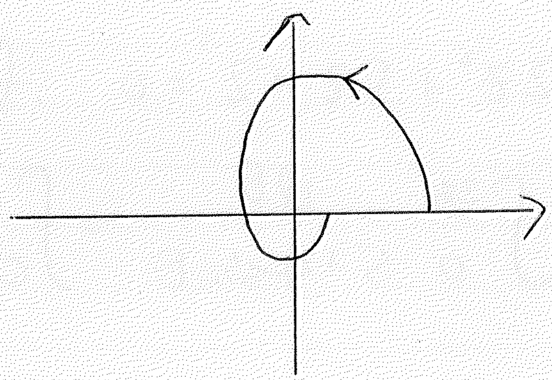
(1a) f has zeros at $z=0$ with order 2.

(1b) f has poles at $z=0$ with order 2.

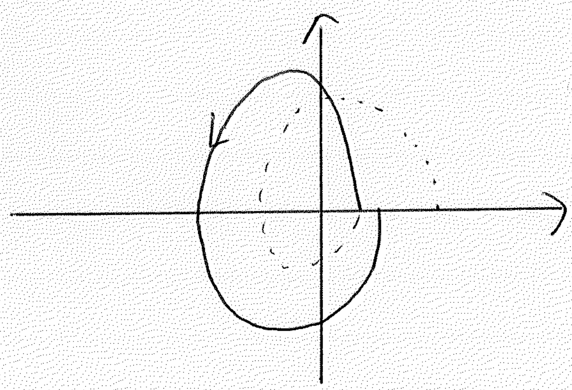
(1c) order of zero at $z=1/2$: 7

order of pole at $z=0$: 3

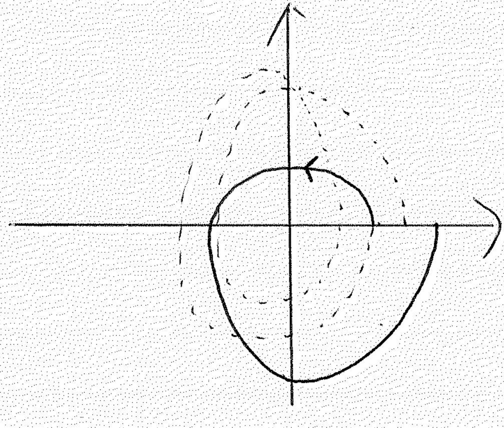
(2) First loop:



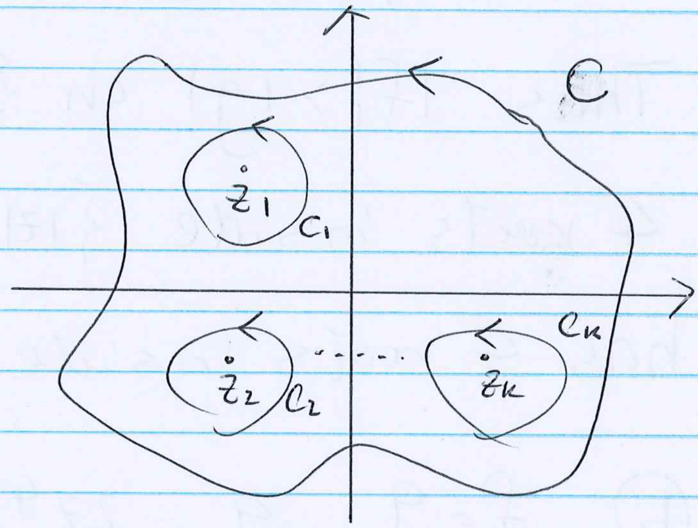
Second loop:



Third loop:



~~(3)~~ ~~(4)~~ (5) Since each zeros are isolated, we can find n small circle centred at $z=z_k$ such that they are disjoint, we denote the circle by C_k.



$$\int_C \frac{zf'}{f} dz = \sum_{k=1}^{n} \int_{C_k} \frac{zf'}{f} dz$$

In C_k, since f has zero ~~has~~ at $z=z_k$ with multiplicity m_k, then f can be expressed by $f=(z-z_k)^{m_k} g_k$ with g being analytic and $g_k(z) \neq 0$ inside C_k, Thus,

$$\int_{C_k} \frac{zf'}{f} = \int_{C_k} \frac{zg_k'}{g_k} + \frac{z m_k}{(z-z_k)} dz$$

Since g_k'/g_k is analytic inside C_k, then

$$\int_{C_k} \frac{zf'}{f} = \int_{C_k} \frac{z m_k}{z-z_k} dz = 2\pi i z_k m_k$$

Therefore $\int_C \frac{zf'}{f} = 2\pi i \sum_{k=1}^{n} m_k z_k$.

(6a) let $f = -5z^4$, $g = z^6 + z^3 - 2z$,

on the circle $\{|z|=1\}$, $|g| \le 1 + 1 + 2$
$= 4$.

$|f| = 5$

Thus $|f| > |g|$ on $\{|z|=1\}$. Since f has 4 roots inside $\{|z|=1\}$, then $z^6 - 5z^4 + z^3 - 2z$ has 4 roots inside $\{|z|=1\}$.

(b) $f = 9$, $g = 2z^4 - 2z^3 + 2z^2 - 2z$.

(c) $f = -4z^3$, $g = z^7 + z - 1$.

(8) let $f_1 = 2z^5$, $g_1 = -6z^2 + z + 1$,

on $\{|z|=2\}$, $|g_1| \le 6 + 1 + 1 = 8 < |f_1| = 64$

$f_1 + g_1$ has 5 roots in $\{|z|=2\}$.

let $f_2 = -6z^2$, $g_2 = 2z^5 + z + 1$

$|g_2| \le 2 + 1 + 1 = 4 < |f_2| = 6$ on $\{|z|=1\}$.

$f_2 + g_2$ has 2 roots in $\{|z|=1\}$

$2z^5 - 6z^2 + z + 1 = 0$ has 3 roots in ~~$\{|z|$~~ the annulus.